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New Subclass Preserving Integral Operator and its properties

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ABSTRACT. In this paper, we introduce a new class of uniformly convex functions defined by a certain fractional calculus operators. The subclass has interesting subclasses like β-uniformly starlike, β-uniformly convex and β-uniformly pre-starlike functions. Properties like coefficient estimates, growth and distortion theorems modified Hadamard product, inclusion property, extreme points, closure theorem and other properties of this class are studied. Lastly, we discuss a class preserving integral operator and the radius of starlikeness, convexity and close-to-convexity for functions in the defined class.

Keywords and Phrases: Fractional derivative, Univalent functions, Uniformly convex function, integral operator, Modified Hadamard Product.
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1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Let $U^* = \{z : 0 < |z| < 1\}$ be the punctured unit disc. Also denote by T the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0, z \in U) \tag{1.2}$$

which are analytic and univalent in U.

For $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ the modified Hadamard product of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

A function $f(z) \in S$ is said to be β-uniformly starlike of order $\alpha, (-1 \leq \alpha < 1), \beta \geq 0$ and all $(z \in U)$, denoted by $\beta - S(\alpha)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta \left|\frac{zf'(z)}{f(z)} - 1\right|. \tag{1.4}$$

A function $f(z) \in S$ is said to be β-uniformly convex of order $\alpha, (-1 \leq \alpha < 1), \beta \geq 0$ and all $(z \in U)$, denoted by $\beta - K(\alpha)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta \left|\frac{zf''(z)}{f'(z)}\right|. \tag{1.5}$$

Notice that, $0 - S(\alpha) = S(\alpha)$ and $0 - K(\alpha) = K(\alpha)$, where $S(\alpha)$ and $K(\alpha)$ are respectively the popular classes of starlike and convex functions of order α $(0 \leq \alpha < 1)$. The classes $\beta - S(\alpha)$ and $\beta - K(\alpha)$ were introduced and studied by Goodman [3]and Minda and Ma [6].

Clearly $f \in \beta - K(\alpha)$ if and only if $zf' \in \beta - S(\alpha)$. Let $\phi(a, c; z)$ be the incomplete beta function defined by

$$\phi(a, c; z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \quad (a \neq -1, -2, -3, \cdots \quad \text{and} \quad c \neq 0, -1, -2, -3, \cdots) \tag{1.6}$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & : \quad k = 0 \\ a(a+1)(a+2)\cdots(a+k-1) & : \quad k \in I\!N \end{cases}$$

We note that $L(a,c)f(z) = \phi(a,b;z) * f(z)$, for $f \in S$ is the Carlson-Shaffer operator [1]. The fractional derivative operator $J_{0,z}^{\mu,\gamma,\eta}$ of a function $f(z)$ is defined as follows.

For $m-1 \leq \mu < m; m \in I\!N$ and $\gamma, \eta \in I\!R$

$$J_{0,z}^{\mu,\gamma,\eta} f(z) = \frac{d^m}{dz^m}\left\{\frac{z^{\mu-\gamma}}{\Gamma(m-\mu)}\int_0^z (z-t)^{m-\mu-1} \right. \\ \left. {}_2F_1(\gamma-\mu, m-\eta; m-\mu; 1-\frac{t}{z})f(t)dt\right\} \tag{1.7}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin with the order

$$f(z) = 0(|z|^r), \quad z \to 0$$

where $r > \max\{0, \gamma-\eta\}-1$ and the multiplicity of $(z-t)^{m-\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$ and is well defined in the unit disc.

Notice that $J_{0,z}^{\mu,\mu,\eta} f(z) = D_{0,z}^{\mu} f(z)$ which is the well known Riemann-Liouville fractional derivative operator [8].

The fractional operator $U_{0,z}^{\mu,\gamma,\eta}$ is defined in terms of $J_{0,z}^{\mu,\gamma,\eta}$ for convenience as follows

$$U_{0,z}^{\mu,\gamma,\eta} = \frac{\Gamma(2-\gamma)\Gamma(2-\mu+\eta)}{\Gamma(2-\gamma+\eta)} z^{\gamma} J_{0,z}^{\mu,\gamma,\eta} f(z) \tag{1.8}$$

$(-\infty < \mu < 1; -\infty < \gamma < 1; \eta \in I\!R^+)$.

Thus,

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = z + \sum_{k=2}^{\infty} \frac{(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} a_k z^k.$$

Note that

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} \frac{\Gamma(2-\gamma)\Gamma(2-\mu+\eta)}{\Gamma(2-\gamma+\eta)} z^{\gamma} J_{0,z}^{\mu,\gamma,\eta} f(z); & 0 \leq \mu < 1 \\ \frac{\Gamma(2-\gamma)\Gamma(2-\mu+\eta)}{\Gamma(2-\gamma+\eta)} z^{\gamma} I_{0,z}^{-\mu,\gamma,\eta}; & -\infty \leq \mu < 0 \end{cases} \tag{1.9}$$

for fractional differential operator $J_{0,z}^{\mu,\gamma,\eta}$ and fractional integral operator $I_{0,z}^{-\mu,\gamma,\eta}$.

Let us now consider another operator $M_{0,z}^{\mu,\gamma,\eta}$ defined using the operators $U_{0,z}^{\mu,\gamma,\eta}$ and the incomplete beta function $\phi(a,b;z)$ as follows.

For real numbers $\mu(-\infty < \mu < 1), \gamma(-\infty < \gamma < 1)$, $\eta \in I\!R^+, a \neq -1, -2, \cdots$, and $c \neq 0, -1, -2, \cdots$ we define the operator $M_{0,z}^{\mu,\gamma,\eta} : S \to S$ by

$$M_{0,z}^{\mu,\gamma,\eta} f(z) = \phi(a,b;z) * U_{0,z}^{\mu,\gamma,\eta} f(z) \tag{1.10}$$

$$= z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} a_k z^k$$

$$= z + \sum_{k=2}^{\infty} h(k) a_k z^k \tag{1.11}$$

for

$$h(k) = \frac{(a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} \tag{1.12}$$

Notice that,

$$M_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} f(z) & \text{if } \ a = c = 1; \quad \mu = \gamma = 0 \\ zf'(z) & \text{if } \ a = c = 1; \quad \mu = \gamma = 1 \end{cases}$$

Consider the subclass $S_{\mu,\gamma,\eta}(\alpha,\beta)$ consisting of functions $f \in S$ and satisfying

$$Re\left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - \alpha \right\} \geq \beta \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| \tag{1.13}$$

$(z \in U, -\infty < \mu < 1; -\infty < \gamma < 1; \eta \in I\!R^+; -1 \leq \alpha < 1; \beta \geq 0; a \neq -1, -2, \cdots; c \neq 0, -1, -2, \cdots)$.

Let $K_{\mu,\gamma,\eta}(\alpha,\beta) = S_{\mu,\gamma,\eta}(\alpha,\beta) \cap T$.

It is also interesting to note that the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$ extend to classes of starlike, convex, β-uniformly starlike, β-uniformly convex and prestarlike function for suitable choice of the parameters $a, c, \mu, \gamma, \eta, \alpha$ and β. For instance;

1. for $a = c = 1; \mu = \gamma = 0$ the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$ reduces to the class of $\beta - S(\alpha)$.

2. For $a = c = 1; \mu = \gamma = 1$ the class reduces to $\beta - K(\alpha)$.

3. For $a = 2 - 2\alpha, c = 1; \mu = \gamma = 0$ the class reduces to β-pre-starlike functions.

Several other classes studied can be derived from $K_{\mu,\gamma,\eta}(\alpha,\beta)$.

2. COEFFICIENT ESTIMATES

Theorem 2.1. *A function $f(z)$ defined by (1.2) is in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$, if and only if*

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k \leq 1-\alpha \tag{2.1}$$

where $0 \leq \alpha < 1; \beta \geq 0, -\infty < \mu < 1, -\infty < \gamma < 1,\ \eta \in I\!R^+, a \neq -1, -2, \cdots$ *and* $c \neq 0, -1, -2, \cdots$.

Proof. Assume (1.2) holds, then we show that $f(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$. Thus, it is suffices to show that

$$\beta \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| - Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - \alpha \right\} \leq 0$$

that is,

$$\beta \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| - Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right\} \leq 1-\alpha.$$

We have

$$\begin{aligned} &\beta \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| - Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right\} \\ &\leq (1+\beta) \left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| \\ &\leq \frac{(1+\beta) \sum_{k=2}^{\infty} (k-1)h(k)a_k}{1 - \sum_{k=2}^{\infty} h(k)a_k}. \end{aligned}$$

This expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k \leq 1-\alpha \tag{2.2}$$

Conversely, we show that a function $f(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$ satisfies inequality (2.1).

Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$ and z be real, then by relation (1.11) and (1.13), we have

$$\frac{1 - \sum_{k=2}^{\infty} kh(k)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} h(k)a_k z^{k-1}} - \alpha \geq \beta \left| \frac{\sum_{k=2}^{\infty} (k-1)h(k)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} h(k)a_k z^{k-1}} \right|.$$

Allowing $z \to 1$ along real axis, we obtain the desired inequality (2.2).

The equality in (2.2) is attained for the extremal function

$$f(z) = z - \frac{(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]h(k)} z^k \quad (k \geq 2). \tag{2.3}$$

□

Corollary 2.2. *Let a function f defined by (1.2) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then*

$$a_k \leq \frac{(1-\alpha)(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}}{[k(1+\beta)-(\alpha+\beta)](a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}}, \quad k \geq 2.$$

Next, we give the growth and distortion theorem for the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$.

Theorem 2.3. *Let the function $f(z)$ defined by (1.2) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then*

$$(2.4)\quad ||M_{0,z}^{\mu,\gamma,\eta}f(z)|-|z|| \leq \frac{c(1-\alpha)(2-\gamma)(2-\mu+\eta)}{2a(\beta-\alpha+2)(2-\gamma+\eta)}|z|^2$$

$$(2.5)\quad ||(M_{0,z}^{\mu,\gamma,\eta}f(z))'|-1| \leq \frac{c(1-\alpha)(2-\gamma)(2-\mu+\eta)}{a(\beta-\alpha+2)(2-\gamma+\eta)}|z|$$

Note that for $a=c=1; \beta=1$, we get the result obtained by G. Murugusundaramoorthy, T. Rosy and M. Darus in [7]. The bounds in (2.4) and (2.5), are attained for the function

$$f(z) = z - \frac{c(1-\alpha)(2-\gamma)(2-\mu+\eta)}{2a(\beta-\alpha+2)(2-\gamma+\eta)}z^2$$

3. CHARACTERIZATION PROPERTY

Theorem 3.1. *Let $\mu,\gamma,\eta \in I\!R$ such that $\mu(-\infty<\mu<1), \gamma(-\infty<\gamma<1), \eta \in I\!R^+, a \neq -1,-2,\cdots$ and $c \neq 0,-1,-2,\cdots$. Also let the function $f(z)$ given by (1.2) satisfy*

$$(3.1)\qquad \sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]}{1-\alpha}h(k)a_k \leq \frac{1}{h(2)}$$

for $-1 \leq \alpha < 1, \beta \geq 0$. Then $M_{0,z}^{\mu,\gamma,\eta}f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$, where $h(k)$ is given by (1.12).

Proof. We have from (1.11)

$$(3.2)\qquad M_{0,z}^{\mu,\gamma,\eta}f(z) = z - \sum_{k=2}^{\infty}h(k)a_k z^k.$$

Under the condition stated in the hypothesis of this theorem, we observe that the function $h(k)$ is a non-increasing function of k for $k \geq 2$, and thus

$$(3.3)\qquad 0 < h(k) \leq h(2) = \frac{2a(2-\gamma+\eta)}{c(2-\gamma)(2-\mu+\eta)}.$$

Therefore, (3.1) and (3.3) yields

$$\sum_{k=2}^{\infty}\frac{k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}h(k)a_k \leq h(2)\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)}h(k)a_k \leq 1.$$

Hence by Theorem 1, we conclude that

$$M_{0,z}^{\mu,\gamma,\eta}f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta).$$

□

Remark. *The equality in (3.1) is attained for the function $f(z)$ defined by*

$$f(z) = z - \frac{c^2(1-\alpha)(2-\gamma)^2(2-\mu+\eta)^2}{4a^2(\beta-\alpha+2)(2-\gamma+\eta)^2} z^2. \tag{3.5}$$

4. RESULTS ON MODIFIED HADAMARD PRODUCT

Theorem 4.1. *For functions $f(z)$ and $g(z)$ defined by (1.2), let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ and $g(z) \in K_{\mu,\gamma,\eta}(\xi,\beta)$. Then*

$$(f * g)(z) \in K_{\mu,\gamma,\eta}(\delta,\beta)$$

where

$$\delta = 1 - \frac{(1+\beta)(1-\alpha)(1-\xi)}{(\beta-\alpha+2)(\beta-\xi+2)h(2) - (1-\alpha)(1-\xi)} \tag{4.1}$$

for $h(2)$ defined by (3.3).

The result is sharp for

$$f(z) = z - \frac{(1-\alpha)}{(\beta-\alpha+2)h(2)} z^2$$

and

$$g(z) = z - \frac{(1-\alpha)}{(\beta-\xi+2)h(2)} z^2$$

Proof. In view of Theorem 2.1 it is sufficient to show that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\delta+\beta)]h(k)}{1-\delta} a_k b_k \leq 1 \tag{4.2}$$

for δ defined by (4.1).

Now, $f(z)$ and $g(z)$ belong to $K_{\mu,\gamma,\eta}(\alpha,\beta)$ and $K_{\mu,\gamma,\eta}(\xi,\beta)$, respectively and so, we have

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{1-\alpha} a_k \leq 1 \tag{4.3}$$

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\xi+\beta)]h(k)}{1-\xi} b_k \leq 1 \tag{4.4}$$

By applying Cauchy-Schwarz inequality to (4.3) and (4.4), we get

$$\sum_{k=2}^{\infty} \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]}}{\sqrt{(1-\alpha)(1-\xi)}} h(k)\sqrt{a_k b_k} \leq 1. \tag{4.5}$$

In view of (4.2) it suffices to show that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\delta+\beta)]h(k)}{1-\delta} a_k b_k \leq \sum_{k=2}^{\infty} \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][(k(1+\beta)-(\xi+\beta)]}}{\sqrt{(1-\alpha)(1-\xi)}} h(k)\sqrt{a_k b_k}$$

or equivalently

$$(4.6)\ \sqrt{a_k b_k} \leq \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]}}{\sqrt{(1-\alpha)(1-\xi)}} \frac{1-\delta}{[k(1+\beta)-(\delta+\beta)]} \quad \text{for} \quad k \geq 2.$$

In view of (4.5) and (4.6) it is sufficient to show that

$$\frac{\sqrt{(1-\alpha)(1-\xi)}}{h(k)\sqrt{[k(1+\beta)-(\alpha+\beta)[k(1+\beta)-(\xi+\beta)]}} \leq \frac{\sqrt{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]}(1-\delta)}{\sqrt{(1-\alpha)(1-\xi)}[k(1+\beta)-(\delta+\beta)]} \quad \text{for} \quad k \geq 2$$

which simplifies to

$$(4.7) \quad \delta \leq 1 - \frac{(1+\beta)(k-1)(1-\alpha)(1-\xi)}{[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\xi+\beta)]h(k)-(1-\alpha)(1-\xi)}$$

where

$$h(k) = \frac{(a)_{k-1}(2-\gamma+\eta)_{k-1}(2)_{k-1}}{(c)_{k-1}(2-\gamma)_{k-1}(2-\mu+\eta)_{k-1}} \quad \text{for} \quad k \geq 2.$$

Notice that $h(k)$ is a decreasing function of k ($k \geq 2$), and thus δ can be chosen as below.

$$\delta = 1 - \frac{(1+\beta)(1-\alpha)(1-\xi)}{(\beta-\alpha+2)(\beta-\xi+2)h(2)-(1-\alpha)(1-\xi)}$$

for $h(2)$ defined by (3.3). This completes the proof. □

Theorem 4.2. *Let the function $f(z)$ and $g(z)$ be defined by (2.1) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $(f*g)(z) \in K_{\mu,\gamma,\eta}(\delta,\beta)$, where*

$$\delta = 1 - \frac{(1+\beta)(1-\alpha)^2}{(\beta-\alpha+2)^2 h(2)-(1-\alpha)^2}$$

for $h(2)$ given by (3.3).

Proof. Substituting $\alpha = \xi$ in the Theorem 4.1 above, the result follows. □

Theorem 4.3. *Let the function $f(z)$ defined by (1.2) be in the class $K_{\mu,\gamma,\eta}(\alpha,\beta)$. Consider*

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \quad \text{for} \quad |b_k| \leq 1.$$

Then $(f * g)(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$.

Proof. Notice that

$$\begin{aligned}
&\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)|a_k b_k| \\
&= \sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k|b_k| \\
&\le \sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]h(k)a_k \\
&\le 1-\alpha \qquad\qquad \text{using Theorem 2.1.}
\end{aligned}$$

Hence $(f * g)(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$. □

Corollary 4.4. *Let the function* $f(z)$ *defined by (1.2) be in the class* $K_{\mu,\gamma,\eta}(\alpha, \beta)$*. Also let* $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ *for* $0 \le b_k \le 1$*. Then* $(f * g)(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$*.*

Next we prove the following inclusion property for functions in the class $K_{\mu,\gamma,\eta}(\alpha, \beta)$.

Theorem 4.5. *Let the functions* $f(z)$ *and* $g(z)$ *defined by (2.1) be in the class* $K_{\mu,\gamma,\eta}(\alpha, \beta)$*. Then the function* $h(z)$ *defined by*

$$h(z) = z - \sum_{k=2}^{\infty}(a_k^2 + b_k^2)z^k$$

is in the class $K_{\mu,\gamma,\eta}(\theta, \beta)$ *where*

$$\theta = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(\beta-\alpha+2)^2 h(2) - 2(1-\alpha)^2}$$

with $h(2)$ *given by (3.3).*

Proof. In view of Theorem 2.1 it is sufficient to show that

$$\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\theta+\beta)]h(k)}{1-\theta}(a_k^2 + b_k^2) \le 1. \tag{4.8}$$

Notice that, $f(z)$ and $g(z)$ belong to $K_{\mu,\gamma,\eta}(\alpha, \beta)$ and so

$$\sum_{k=2}^{\infty}\left[\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}\right]^2 a_k^2 \le \left[\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}a_k\right]^2 \le 1 \tag{4.9}$$

$$\sum_{k=2}^{\infty}\left[\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}\right]^2 b_k^2 \le \left[\sum_{k=2}^{\infty}\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}b_k\right]^2 \le 1\ . \tag{4.10}$$

Adding (4.9) and (4.10), we get

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)}\right]^2 (a_k^2 + b_k^2) \le 1. \tag{4.11}$$

Thus (4.8) will hold if

$$\frac{[k(1+\beta)-(\theta+\beta)]}{1-\theta} \le \frac{1}{2}\frac{h(k)[k(1+\beta)-(\alpha+\beta)]^2}{(1-\alpha)^2}.$$

That is, if

$$\theta \le 1 - \frac{2(1+\beta)(k-1)(1-\alpha)^2}{[k(1+\beta)-(\alpha+\beta)]^2 h(k) - 2(1-\alpha)^2}\ . \tag{4.12}$$

Notice that, θ can be further improved by using the fact that $h(k) \le h(2)$ for $k \ge 2$. Therefore,

$$\theta = 1 - \frac{2(1+\beta)(1-\alpha)^2}{(\beta-\alpha+2)^2 h(2) - 2(1-\alpha)^2}$$

where $h(2)$ is given by (3.3). □

5. INTEGRAL TRANSFORM OF THE CLASS $K_{\mu,\gamma,\eta}(\alpha,\beta)$

For $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ we define the integral transform

$$L_\lambda(f)(z) = \int_0^1 \frac{\lambda(t) f(tz)}{t} dt$$

where $\lambda(t)$ is real valued, non-negative weight function normalized such that $\int_0^1 \lambda(t)dt = 1$. Note that, $\lambda(t)$ have several special interesting definitions. For instance, $\lambda(t) = (1+c)t^c, c > -1$, for which L_λ is known as the Bernardi operator. For

$$\lambda(t) = \frac{2^\delta}{\Gamma(\delta)} t(\log \frac{1}{t})^{\delta-1}, \quad \delta \ge 0 \tag{5.1}$$

we get the integral operator introduced by Jung, Kim and Srivastava [8].

Let us consider the function

$$\lambda(t) = \frac{(c+1)^\delta}{\Gamma(\delta)} t^c(\log \frac{1}{t})^{\delta-1}, \quad c > -1, \quad \delta \ge 0. \tag{5.2}$$

Notice that for $c = 1$ we get the integral operator introduced by Jung, Kim and Srivastava.

We next show that the class is closed under $L_\lambda(f)$ for $\lambda(t)$ given by (5.2).

Theorem 5.1. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$.*

Proof. By using the definition of $L_\lambda(f)$, we have

$$\begin{aligned} L_\lambda(f) &= \frac{(c+1)^\delta}{\Gamma(\delta)} \int_0^1 \frac{t^c(\log \frac{1}{t})^{\delta-1} f(tz)}{t} dt \\ &= \frac{(c+1)^\delta}{\Gamma(\delta)} \int_0^1 (\log \frac{1}{t})^{\delta-1} t^c \left(z - \sum_{k=2}^{\infty} a_k t^{k-1} z^k \right) dt. \end{aligned} \tag{5.3}$$

Simplifying by using the definition of gamma function, we get

$$L_\lambda(f) = z - \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right)^{\delta} a_k z^k. \tag{5.4}$$

Now $L_\lambda(f) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ if

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)} \left(\frac{c+1}{c+k}\right)^{\delta} a_k \le 1. \tag{5.5}$$

Also by Theorem 2.1 we have $f \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)} a_k \le 1. \tag{5.6}$$

Thus, in view of (5.5) and (5.6) and the fact that $\left(\frac{c+1}{c+k}\right) < 1$ for $k \ge 2$, (5.5) holds true. Therefore, $L_\lambda(f) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ and the proof is complete. □

6. EXTREME POINTS OF $K_{\mu,\gamma,\eta}(\alpha,\beta)$

Theorem 6.1. *Let*

$$f_1(z) = z \tag{6.1}$$

and

$$f_k(z) = z - \frac{(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]h(k)} z^k, \quad (k \ge 2). \tag{6.2}$$

Then $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

where $\lambda_k \ge 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Let $f(z)$ be expressible in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

Then

$$f(z) = z - \sum_{k=2}^{\infty} \frac{(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]h(k)} \lambda_k z^k.$$

Now,

$$\sum_{k=2}^{\infty} \frac{(1-\alpha)\lambda_k}{[k(1+\beta)-(\alpha+\beta)]} \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)} = \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \le 1.$$

Therefore, $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$.

Conversely, suppose that $f(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$. Thus,

$$a_k \leq \frac{(1-\alpha)}{[k(1+\beta)-(\alpha+\beta)]h(k)} \quad (k \geq 2).$$

Setting

$$\lambda_k = \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)} a_k \quad (k \geq 2)$$

and $\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$, we get

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

This completes the proof. □

7. CLOSURE THEOREM

Theorem 7.1. *Let the function $f_j(z)$ defined by (2.1) be in the class $K_{\mu,\gamma,\eta}(\alpha, \beta)$. Then the function $h(z)$ defined by*

$$h(z) = z - \sum_{k=2}^{\infty} e_k z^k \text{ belongs to } K_{\mu,\gamma,\eta}(\alpha, \beta)$$

where $f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k, \quad j = 1, 2, \cdots, \ell,$ and

$$e_k = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j} \qquad (a_{k,j} \geq 0).$$

Proof. Since $f_j(z) \in K_{\mu,\gamma,\eta}(\alpha, \beta)$, in view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)} a_{k,j} \leq 1. \tag{7.1}$$

Now,

$$\begin{aligned} \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z) &= \frac{1}{\ell} \sum_{j=1}^{\ell} \left(z - \sum_{k=2}^{\infty} a_{k,j} z^k \right) \\ &= z - \sum_{k=2}^{\infty} e_k z^k \end{aligned}$$

where $e_k = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j}$.

Notice that,

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(\alpha+\beta)]h(k)}{(1-\alpha)} \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j} \leq 1, \quad \text{using (7.1)}.$$

Thus, $h(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. □

8. RADIUS OF STARLIKENESS, CONVEXITY AND CLOSE-TO-CONVEXITY

Theorem 8.1. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is starlike of order $s, 0 \leq s < 1$ in $|z| < R_1$ where*

$$R_1 = \inf_k \left[\frac{(1-s)[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)(k-s)} \right]^{\frac{1}{(k-1)}}. \tag{8.1}$$

Proof. $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is said to be starlike of order $s, 0 \leq s < 1$, if and only if

$$Re \left\{ \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} \right\} > s \tag{8.2}$$

or equivalently

$$\left| \frac{z(M_{0,z}^{\mu,\gamma,\eta} f(z))'}{M_{0,z}^{\mu,\gamma,\eta} f(z)} - 1 \right| < 1 - s.$$

With fairly straight forward calculations, we get

$$|z|^{k-1} \leq \frac{(1-s)[k(1+\beta)-(\alpha+\beta)]}{(1-\alpha)(k-s)}, \quad k \geq 2.$$

Setting $R_1 = |z|$, the result follows. □

Next, we state the radius of convexity using the fact that f is convex, if and only if zf' is starlike. We omit the proof of the following theorems as the results can be easily derived.

Theorem 8.2. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is convex of order $c, 0 \leq c < 1$ in $|z| < R_2$ where*

$$R_2 = \inf_k \left[\frac{(1-c)[k(1+\beta)-(\alpha+\beta)]}{k(1-\alpha)(k-c)} \right]^{\frac{1}{(k-1)}}.$$

Theorem 8.3. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $M_{0,z}^{\mu,\gamma,\eta} f(z)$ is close-to-convex of order $r, 0 \leq r < 1$ in $|z| < R_3$ where*

$$R_3 = \inf_k \left[\frac{(1-r)[k(1+\beta)-(\alpha+\beta)]}{k(1-\alpha)} \right]^{\frac{1}{(k-1)}}.$$

Thoerem 8.4. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)$ is starlike of order $p, 0 \le p < 1$ in $|z| < R_4$ where*

$$R_4 = \inf_k \left[\frac{(1-p)[k(1+\beta)-(\alpha+\beta)]h(k)(c+k)^\delta}{(1-\alpha)(k-p)(c+1)^\delta}\right]^{\frac{1}{(k-1)}}.$$

Theorem 8.5. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)$ is convex of order $q, 0 \le q < 1$ in $|z| < R_5$ where*

$$R_5 = \inf_k \left[\frac{(1-q)[k(1+\beta)-(\alpha+\beta)]h(k)(c+k)^\delta}{k(1-\alpha)(k-q)(c+1)^\delta}\right]^{\frac{1}{(k-1)}}.$$

Theorem 8.6. *Let $f(z) \in K_{\mu,\gamma,\eta}(\alpha,\beta)$. Then $L_\lambda(f)$ is close-to-convex of order $m, 0 \le m < 1$ in $|z| < R_6$ where*

$$R_6 = \inf_k \left[\frac{(1-m)[k(1+\beta)-(\alpha+\beta)]h(k)(c+k)^\delta}{k(1-\alpha)(c+1)^\delta}\right]^{\frac{1}{(k-1)}}.$$

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